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Global classical solutions for a free-boundary problem modeling combustion of solid propellants

D. Pierotti *, M. Verri

Dipartimento di Matematica, Politecnico di Milano, piazza Leonardo da Vinci 32, 20133 Milano, Italy

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Abstract

We study a free-boundary problem for the heat equation in one space dimension, describing the burning of a semi-infinite adiabatic solid propellant subjected to external thermal radiation (typically, a laser). The model includes the presence on the moving solid–gas interface (the free boundary) of heat release, due both to propellant degradation and conductive heat feedback from the gas phase reactions. The pyrolysis law and the flame submodel, relating burning rate to the boundary temperature and the heat feedback, respectively, satisfy general and physically significant conditions. We prove existence and uniqueness of a classical solution, local in time, for continuous initial thermal profiles. In addition, if the initial datum is exponentially bounded at infinity, we derive the main result of existence in the large and some uniform bounds for the solution.

Keywords: Free-boundary problem; Heat equation; Combustion

1. Introduction

Let the half-line $-\infty < y < s(t)$ be filled with a solid body burning under the influence of external thermal irradiation of the solid–gas interface, which occurs at the "surface" $y = s(t)$. The front $s(t)$ is propagating into the solid and it is assumed that the heat flux $q = q_c + q_g$ entering the condensed phase at the boundary, due to solid degradation reactions (q_c) and heat feedback from the gas phase (q_g), is prescribed as a function of the burning rate $r = -\dot{s}(t)$. If chemical

* Corresponding author.
E-mail address: darpie@mate.polimi.it (D. Pierotti).

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reactions in the solid are neglected, the concentrated surface gasification resorts to some given pyrolysis relationship for r in terms of the boundary temperature ξ, say $r = r(\xi)$. The initial thermal profile of the body is prescribed and the radiant flux intensity I, heat capacity c and thermal conductivity k are all positive constants. The problem is to determine the temperature $T(y,t)$ of the body and the free boundary $s(t)$, subject to the following governing equations and conditions:

$$c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial y^2}, \quad y < s(t),\ t > 0, \tag{1.1a}$$

$$\frac{\partial T}{\partial y}(-\infty, t) = 0, \quad t > 0, \tag{1.1b}$$

$$k\frac{\partial T}{\partial y}\big(s(t), t\big) = I + q\Big(r\Big(T\big(s(t), t\big)\Big)\Big), \quad t > 0, \tag{1.1c}$$

$$T(y, 0) = T_{\mathrm{in}}(y), \quad y \leqslant 0, \tag{1.1d}$$

$$-\dot{s}(t) = r\Big(T\big(s(t), t\big)\Big), \quad t > 0, \tag{1.1e}$$

$$s(0) = 0. \tag{1.1f}$$

The above model describes radiation-driven unsteady combustion of solid energetic materials. This area is of fundamental but also practical interest in connection with combustion instabilities in rocket engine applications, numerical simulations of time-dependent processes of solid propellant deflagration, such as radiative ignition and deradiative extinguishment, etc. (see, for example, the review by DeLuca [4]). The classical theory, originated from the pioneering ideas of Zel'dovich [16] and Novozhilov [10], relies on the key modeling assumptions of quasiplanarity of all spatial variations and quasisteadiness of the gas phase [2,6,11,15]. Within this framework we are justified in approximating the propellant as a semi-infinite solid and the burning surface as a very thin layer of the condensed phase. A number of extra assumptions are incorporated into the model, including homogeneous and chemically inert propellant, absence of conductive and radiative heat losses, one-step irreversible gasification processes, fully opaque burning surface. However, we make no specific assumptions about the pyrolysis law and the flame submodel as long as some natural and physically meaningful conditions are satisfied by the related functions $r(\xi)$ and $q_{\mathrm{g}}(r)$, respectively. Typically, we may suppose $r(\xi)$ is a function vanishing below some given ignition temperature T_0 and increasing for $\xi > T_0$, as in general observed for one-step, irreversible gasification processes. Specific examples include the standard Arrhenius law or similar [5]. Also, we may assume the conductive heat feedback $q_{\mathrm{g}} = q_{\mathrm{g}}(r)$ to be a function vanishing as $r \to \infty$ (flame blow-off) and as $r \to 0$ (no burning) and positive for $r > 0$ (gas phase heats up condensed phase during burning), as observed for one-phase, laminar, nonviscous, low-subsonic, thermal flames. As a matter of fact, these properties are possessed by most of the available flame models, valid for a variety of energetic materials as well as for a wide set of operating conditions [4]. For an illustration, consider $q_{\mathrm{g}}(r) = q_0(1 - e^{-\tau_0 r^2})/r$, where q_0 and τ_0 are positive constants.

One of the main physical and mathematical features of the model (1.1) is the occurrence of the heat feedback from the gas to the condensed phase, accounted for by including the *non-monotonic* term q_{g} in the burning surface flux condition. The gasless case (i.e., $q_{\mathrm{g}} \equiv 0$) with linear surface heat release due to chemical reactions (i.e., $q_{\mathrm{c}} = hr$, where the latent heat h is constant) and in the absence of external radiant source ($I = 0$) amounts to a well-known mathematical model which has been considered by several authors to describe a variety of exothermic phase transition-type processes, such as solidification with undercooling, rapid crystallization in thin films, etc. The latter model has been studied extensively by Frankel and Roytburd [9, and references therein]. Rather, in the outlined context of solid-propellant combustion we refer to, the solid surface temperature adjustments due to solid–gas coupling are expected to have a large influence on the burning phenomenon [6], so a proper choice of a nonzero q_g is of critical importance for formulating a correct model of the underlying physical process.

A point of interest in our investigation is the global stability analysis of traveling waves solutions of the free-boundary-value problem (1.1), that is, thermal profiles $T(y,t) = \overline{T}(y + \bar{r}t)$ corresponding to a constant speed $\bar{r}$ of the front. Such uniformly propagating profiles are easy to find, namely,

$$\overline{T}(y + \bar{r}t) = T_1 + \big(\overline{T}(0) - T_1\big)\exp\left(\frac{c\bar{r}}{k}(y + \bar{r}t)\right), \tag{1.2}$$

where T_1 is the value of the temperature "far ahead" of the flame, $\overline{T}(0)$ is a constant greater than T_0 and $\bar{r} = r(\overline{T}(0))$. Condition (1.1c), when applied to (1.2), shows that $\bar{r}$ is a positive root of the "eigenvalue" equation

$$c\big(\overline{T}(0) - T_1\big) - \frac{I}{\bar{r}} = \frac{q(\bar{r})}{\bar{r}}. \tag{1.3}$$

In what follows the existence of a unique eigenvalue $\bar{r}$ is assumed. For instance, the transcendental equation (1.3) for $\bar{r}$ has a unique solution when $q(r)/r$ is a decreasing function for $r > 0$.

Stability properties of the wave $\overline{T}(y + \bar{r}t)$ have been investigated under *smooth* and *small* exponentially bounded disturbances at $t = 0$ by using the principle of linearized stability [12,14]. The question then arises as to when an *arbitrary* initial profile converges to the traveling wave as time goes to infinity. The large-time asymptotic behavior of solutions and related questions have been discussed elsewhere [13] by using an invariance technique for which global solvability and regularity of the solution for *continuous* data are crucial (see, e.g., the discussion in [1]). Therefore the present paper is mainly devoted to the preliminary study of existence and uniqueness of global-in-time classical solutions to problem (1.1). It should be pointed out that the essential requirement that enables the gasless model to sustain all-time, and even uniformly bounded, dynamical evolution is the extra assumption of a positive lower bound on r, often referred to as the "ignition velocity." See [7,8]. This assumption could be removed provided a damping term $-\alpha T$ is introduced into the energy conservation equation (1.1a). In contrast, our model (1.1) applies to adiabatic solid (no volumetric heat loss takes place) and it may well happen that r vanishes during the time evolution. The novelties in both the structure of the heat flux function q and the possible degeneracy of the burning rate function r can be treated mathematically by means of classical techniques, which however require nontrivial modifications in order to state the desired properties of the solution in our setting.

The paper is organized as follows. After having the model rewritten in normalized variables (Section 2), local existence and uniqueness are proved in Section 3 for arbitrary continuous data. The analysis is based upon two suitable representations of the solution involving single- and double-layer heat potentials. The global existence and uniform bounds of the solution and its space derivative are then obtained in Section 4 under some restriction on the initial temperature, namely, exponential boundedness at infinity.

2. The normalized model

We shall begin by formulating our model in a more convenient way. The initial thermal profile $T_{\mathrm{in}}(y)$ is postulated to decay at infinity towards some constant value (the ambient temperature) not greater than the surface temperature at ignition. Then, considering the traveling wave (1.2), where $T_1 = T_{\mathrm{in}}(-\infty) \leqslant T_0$, we switch to dimensionless space and time variables by the solid convective–diffusive scaling

$$y_1 = \frac{c\bar{r}}{k} y, \qquad t_1 = \frac{c\bar{r}^2}{k} t, \qquad s_1(t_1) = \frac{c\bar{r}}{k} s(t)$$

and define the normalized temperature U, pyrolysis velocity R and overall heat flux Q, where

$$U(y_1, t_1) = \frac{T(y,t) - T_1}{\overline{T}(0) - T_1}, \qquad R(U) = \frac{r(T)}{\bar{r}}, \qquad Q(U) = \frac{I + q(r)}{c\bar{r}(\overline{T}(0) - T_1)}.$$

Thus the normalized traveling wave is $U = e^{y_1 + t_1}$ and both rate $R(1)$ and heat flux $Q(1)$ at the normalized traveling wave front $y_1 = -t_1$ become unitary, see Eq. (1.3). Omitting the subscript 1 in y_1, t_1 and s_1, we obtain

$$U_t = U_{yy}, \quad y < s(t),\ t > 0, \tag{2.1a}$$

$$U_y(-\infty, t) = 0, \quad t > 0, \tag{2.1b}$$

$$U_y\big(s(t), t\big) = Q\big(U\big(s(t), t\big)\big), \quad t > 0, \tag{2.1c}$$

$$U(y, 0) = U_{\mathrm{in}}(y), \quad y \leqslant 0, \tag{2.1d}$$

$$-\dot{s}(t) = R\big(U\big(s(t), t\big)\big), \quad t > 0, \tag{2.1e}$$

$$s(0) = 0. \tag{2.1f}$$

Throughout the paper we assume that:

H1. $R(u)$ and $Q(u)$ are twice differentiable nonnegative functions such that $R, |R'|, |R''|, Q, |Q'|, |Q''| \leqslant M$, and

H2. $U_{\mathrm{in}}(y)$ is a continuous function such that $U_{\mathrm{in}}(-\infty) = 0$ and $|U_{\mathrm{in}}| \leqslant M$ for some constant $M > 0$.

Remark 2.1. For the sake of mathematical generality we have considered quite general functions R and Q. Actually, from the physical viewpoint it should be required, additionally: $R(1) = Q(1) = 1$; $R(\xi) = 0$ for $\xi \leqslant \xi_0$ and $R(\xi)$ increasing for $\xi > \xi_0$ for some value ξ_0 (the normalized ignition temperature), $0 < \xi_0 < 1$; $Q(\xi) = Q_0$ for $\xi \leqslant \xi_0$, $Q(\xi) > Q_0$ for $\xi > \xi_0$ for some value Q_0 (the normalized radiant intensity), $0 < Q_0 < 1$.

A *solution* (s, U) of problem (2.1) in the time interval $[0, \gamma]$, where $\gamma > 0$, is a pair of functions $s = s(t)$ and $U = U(y, t)$ such that

(i) $s \in \mathcal{C}^1([0, \gamma])$ and $s(0) = 0$;
(ii) U satisfies (2.1a)–(2.1d) for this s in the sense that

$$U \in \mathcal{C}^0(\overline{\Omega}_\gamma), \qquad U_{yy}, U_t \in \mathcal{C}^0(\Omega_\gamma), \qquad U_y \in \mathcal{C}^0(\Omega_\gamma \cup \Gamma_\gamma),$$

where $\Omega_\gamma = \{(y, t)\colon\ y < s(t),\ 0 < t \leqslant \gamma\}$, $\Gamma_\gamma = \{(s(t), t)\colon\ 0 < t \leqslant \gamma\}$;
(iii) s and U satisfy (2.1e).

A pair (s, U) is a *global solution* of problem (2.1) if γ is infinite.

Remark 2.2. If the free-boundary problem (2.1) is transformed to one in a fixed domain by the change of spatial variable $x = y - s(t)$, the field equation then involves a convective term, whose coefficient is a nonlocal function of the front temperature:

$$\begin{aligned}
&\vartheta_t = \vartheta_{xx} - R\big(\vartheta(0,t)\big)\vartheta_x, \quad x < 0,\ t > 0,\\
&\vartheta_x(-\infty,t) = 0, \quad t > 0,\\
&\vartheta_x(0,t) = Q\big(\vartheta(0,t)\big), \quad t > 0,\\
&\vartheta(x,0) = U_{\mathrm{in}}(x), \quad x \leqslant 0,
\end{aligned} \tag{2.2}$$

where $\vartheta(x,t)$ is the temperature in the moving frame. Sometimes this viewpoint is of computational advantage and it will be adopted by us to show Proposition 4.1.

Suppose now that (2.1) possesses a solution and let

$$E(z,\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-z^2/4\tau}, \quad -\infty < z < +\infty,\ \tau > 0,$$

be the one-dimensional heat kernel. Then, integrating the identity

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\big(E(y-z,t-\tau)U(z,\tau)\big)\\
&\quad= \frac{\partial}{\partial z}\left(E(y-z,t-\tau)\,U_z(z,\tau) - U(z,\tau)\,\frac{\partial}{\partial z}E(y-z,t-\tau)\right)
\end{aligned}$$

over the domain $z < s(\tau)$, $0 < \tau < t$, we obtain the representation

$$\begin{aligned}
U(y,t) = {}&\int_{-\infty}^{0} E(y-z,t)U_{\mathrm{in}}(z)\,dz + \int_0^t E\big(y-s(\tau),t-\tau\big)F\big(U\big(s(\tau),\tau\big)\big)\,d\tau\\
&+ \int_0^t E_z\big(y-s(\tau),t-\tau\big)U\big(s(\tau),\tau\big)\,d\tau,
\end{aligned} \tag{2.3}$$

where

$$F(\xi) = Q(\xi) - \xi\, R(\xi). \tag{2.4}$$

Letting $y \uparrow s(t)$, the theorem on discontinuity of the heat potential of a double layer [3, Section 14.2] yields

$$\begin{aligned}
U\big(s(t),t\big) = 2&\int_{-\infty}^{0} E\big(s(t)-z,t\big)U_{\mathrm{in}}(z)\,dz\\
&+ 2\int_0^t E\big(s(t)-s(\tau),t-\tau\big)F\big(U\big(s(\tau),\tau\big)\big)\,d\tau\\
&+ 2\int_0^t E_z\big(s(t)-s(\tau),t-\tau\big)U\big(s(\tau),\tau\big)\,d\tau.
\end{aligned} \tag{2.5}$$

To this we add condition (2.1e) rewritten in integral form

$$s(t) = -\int_0^t R\big(U\big(s(\tau), \tau\big)\big)\, d\tau. \tag{2.6}$$

Substituting $s(t)$ from (2.6) into expression (2.5), we see that the *boundary temperature* $v(t) = U(s(t), t)$ must be a solution, if any, of the integral equation

$$\begin{aligned} v(t) = 2\int_{-\infty}^0 E\big(s(t) - z, t\big)U_{\text{in}}(z)\, dz + 2\int_0^t E\big(s(t) - s(\tau), t - \tau\big)F\big(v(\tau)\big)\, d\tau \\ + 2\int_0^t E_z\big(s(t) - s(\tau), t - \tau\big)v(\tau)\, d\tau, \end{aligned} \tag{2.7}$$

where $s(t) = -\int_0^t R(v(\tau))\, d\tau$. This is the unsteady analog of the eigenvalue equation (1.3) and, accordingly, we have established that the distribution of temperature within the solid is determined exclusively by the temperature of its lateral boundary, which is an unknown of the problem.

Our first result is an immediate consequence of formula (2.3) in agreement with physical reality. In the absence of heat exchange far ahead of the flame, Eq. (2.1b), we expect that the temperature at the cold end should remain constant over time. This is proved in the following lemma.

Lemma 2.1. *For the solution U of* (2.1), $\lim_{y\to-\infty} U(y, t) = 0$ *for any* $t > 0$.

Proof. Let $t > 0$ be fixed. We write $U(y, t) = J_1 + J_2 + J_3$, where J_1, J_2 and J_3 are the three terms on the right-hand side of (2.3), respectively. Consider the integral J_1: for $\varepsilon > 0$ it follows, from $U_{\text{in}}(-\infty) = 0$, that there exists a positive constant K such that $|U_{\text{in}}(z)| < \varepsilon$ for $z < -K$. Consequently, we have

$$|J_1| \leqslant \varepsilon \int_{-\infty}^{-K} E(y - z, t)\, dz + M\int_{-K}^0 E(y - z, t)\, dz.$$

Hence,

$$|J_1| \leqslant \varepsilon + M\int_y^{y+K} E(z, t)\, dz < 2\varepsilon$$

for all sufficiently large y. For J_2, as

$$0 < E\big(y - s(\tau), t - \tau\big) \leqslant \frac{1}{\sqrt{4\pi(t - \tau)}},$$

it follows $J_2 \to 0$ as $y \to -\infty$. Likewise, we see that $J_3 \to 0$ as $y \to -\infty$ since

$$\big|E_z\big(y - s(\tau), t - \tau\big)\big| = \frac{Ae^{-A}}{\sqrt{\pi(t - \tau)}}\big|y - s(\tau)\big|^{-1} \leqslant \frac{C}{\sqrt{\pi(t - \tau)}}\big|y - s(t)\big|^{-1},$$

where $A = (y - s(\tau))^2/4(t-\tau)$, taking into account inequalities $Ae^{-A} \leqslant C$ (a positive constant) and $|y - s(t)| \leqslant |y - s(\tau)|$ for all y with $y < s(t)$ and $0 \leqslant \tau < t$. $\square$

Equation (2.7) will be used later to prove uniqueness and other properties of the solution (s, U). Unfortunately, this boundary integral technique is not suitable for the existence treatment because, due to the presence of the heat potential of the double layer in the representation (2.3), one cannot check the flux boundary condition (2.1c) unless the initial datum is differentiable. To avoid this difficulty, a working approach for existence is to construct a candidate solution where U is represented by a combination of single layer potentials. In fact, as described in the next section, imposing both temperature and heat flux conditions at the lateral boundary yields a system of two integral equations to which a fixed point technique can be applied successfully. Then, checking to see whether we have really found a solution to (2.1) looks now straightforward, since the flux condition is satisfied automatically.

3. The local solution

In this section we prove the existence and uniqueness of a local solution to problem (2.1). As remarked in the introduction, our result is valid for arbitrary continuous data and without extra assumptions on the heat flux and burning rate functions apart from reasonable regularity. We stress that a similar result has been proved in [9] for the "gasless case" (with strictly positive initial data) by solving iteratively a system of integral equations which follow by a standard representation of the solution in terms of heat potentials. In our approach, we introduce a *modified* representation which depends on a small (positive) parameter in the single-layer potential; roughly speaking, the parameter is chosen to compensate the loss of contraction rate due to the $1/\sqrt{t}$ singularity in the potential density. Then, by writing the system of integral equations as a fixed point equation in a suitable Banach space, we solve it (for small enough value of the parameter) by applying the contraction mapping principle.

Theorem 3.1. *Under assumptions* H1 *and* H2*, there exists $\gamma > 0$ such that problem* (2.1) *has a unique solution in the interval* $[0, \gamma]$.

The starting point is the representation of U in the form

$$U(y,t) = V^0(y,t) + 2\int_0^t E\big(y - s(\tau), t-\tau\big)\big[Q\big(\hat{\theta}(\tau)\big) - \varepsilon\hat{\varphi}(\tau)\big]\,d\tau, \tag{3.1}$$

where the constant $\varepsilon > 0$ will be chosen later,

$$s(t) = -\int_0^t R\big(\hat{\theta}(\tau)\big)\,d\tau \tag{3.2}$$

and

$$V^0(y,t) = \int_{-\infty}^0 E(y-z,t)U_{\mathrm{in}}(z)\,dz. \tag{3.3}$$

The unknown densities $\hat{\theta}(t)$ and $\hat{\varphi}(t)$ may be determined by imposing the boundary conditions

$$\lim_{y\uparrow s(t)} U(y,t) = \hat{\theta}(t) \tag{3.4}$$

and

$$\lim_{y\uparrow s(t)} U_y(y,t) = Q\big(\hat{\theta}(t)\big). \tag{3.5}$$

In fact, on the basis of the discontinuity properties of the single-layer potential, it is clear that, if (3.1) holds and the densities $\hat{\theta}$ and $\hat{\varphi}$ belong to appropriate function spaces [3, Chapter 14], then they must satisfy the pair of integral equations

$$\hat{\theta}(t) = V^0\big(s(t),t\big) + 2\int_0^t E\big(s(t)-s(\tau),t-\tau\big)\big[Q\big(\hat{\theta}(\tau)\big) - \varepsilon\hat{\varphi}(\tau)\big]\,d\tau,$$

$$\hat{\varphi}(t) = \varepsilon^{-1}V_y^0\big(s(t),t\big) + 2\int_0^t E_z\big(s(t)-s(\tau),t-\tau\big)\big[\varepsilon^{-1}Q\big(\hat{\theta}(\tau)\big) - \hat{\varphi}(\tau)\big]\,d\tau. \tag{3.6}$$

Notice that $\hat{\theta}$ also appears in the first argument of the functions E and E_z through the *free boundary* s. We write these equations more compactly as

$$\mathbf{v} = \mathbf{K}_\varepsilon \mathbf{v} \tag{3.7}$$

by introducing the vector-valued function $\mathbf{v}(t) = (\hat{\theta}(t), \hat{\varphi}(t))$ and operator $\mathbf{K}_\varepsilon = (K_{\varepsilon 1}, K_{\varepsilon 2})$, where $K_{\varepsilon 1}$ and $K_{\varepsilon 2}$ denote the right-hand sides of (3.6).

In view of the singular behavior of $\hat{\varphi}$ at $t = 0$ the following weighted space of continuous functions is convenient to prove the existence of a unique fixed point of the operator $\mathbf{K}_\varepsilon$. We denote by $\mathcal{V}_\gamma$, $\gamma > 0$, the Banach space of functions $\mathbf{v} = (\hat{\theta}, \hat{\varphi}) \in \mathcal{C}^0([0,\gamma]) \oplus \mathcal{C}^0((0,\gamma])$ such that

$$\|\mathbf{v}\|_\gamma = \sup_{0<t<\gamma}\big|\hat{\theta}(t)\big| + \sup_{0<t<\gamma}\big|t^{1/2}\hat{\varphi}(t)\big| < \infty.$$

Lemma 3.1. *Fix $\varepsilon \in (0, \pi^{-1/2})$ and $\gamma \in (0, \gamma_1)$, where $\gamma_1^{1/2} = 2M^{-1}(\pi^{-1/2} - \varepsilon)$. Then there exists $\delta_1 > 0$ such that $\mathbf{K}_\varepsilon \mathcal{B}_\gamma(\delta) \subset \mathcal{B}_\gamma(\delta)$ for any $\delta \geqslant \delta_1$, where $\mathcal{B}_\gamma(\delta) = \{\mathbf{v} \in \mathcal{V}_\gamma\colon \|\mathbf{v}\|_\gamma \leqslant \delta\}$.*

Proof. We begin our estimates with $K_{\varepsilon 1}$. From (3.3) and the normalization property of the heat kernel it follows immediately that

$$\big|V^0\big(s(t),t\big)\big| \leqslant M$$

for $t \geqslant 0$. Further, for $\mathbf{v} = (\hat{\theta}, \hat{\varphi}) \in \mathcal{V}_\gamma$ and for all $0 \leqslant t \leqslant \gamma$ we have

$$\begin{aligned}
&\left|\int_0^t E\big(s(t)-s(\tau),t-\tau\big)\big[Q\big(\hat{\theta}(\tau)\big) - \varepsilon\hat{\varphi}(\tau)\big]\,d\tau\right| \\
&\quad\leqslant (4\pi)^{-1/2}\left(\int_0^t (t-\tau)^{-1/2} Q\big(\hat{\theta}(\tau)\big)\,d\tau + \varepsilon\int_0^t (t-\tau)^{-1/2}\big|\hat{\varphi}(\tau)\big|\,d\tau\right) \\
&\quad\leqslant \pi^{-1/2}Mt^{1/2} + 2^{-1}\pi^{1/2}\varepsilon \sup_{0<\tau<\gamma}\big|\tau^{1/2}\hat{\varphi}(\tau)\big|,
\end{aligned}$$

where we have used $\int_0^t (t-\tau)^{-1/2}\tau^{-1/2}\,d\tau = \pi$. Thus, we see that

$$\big|(K_{\varepsilon 1}\mathbf{v})(t)\big| \leqslant M + 2\pi^{-1/2}M\gamma^{1/2} + \pi^{1/2}\varepsilon\|\mathbf{v}\|_\gamma. \tag{3.8}$$

Consider now $K_{\varepsilon 2}$. We have

$$\begin{aligned}\big|V_y^0\big(s(t),t\big)\big| &\leqslant \int_{-\infty}^{0} \frac{|s(t)-z|}{2t} E\big(s(t)-z,t\big)\big|U_{\mathrm{in}}(z)\big|\,dz \\ &\leqslant \pi^{-1/2}Mt^{-1/2} \int_{s(t)/\sqrt{4t}}^{+\infty} |\lambda| e^{-\lambda^2}\,d\lambda \leqslant \pi^{-1/2}Mt^{-1/2}.\end{aligned}$$

Next, we write

$$E_z\big(s(t)-s(\tau),t-\tau\big) = -\frac{s(t)-s(\tau)}{2(t-\tau)} E\big(s(t)-s(\tau),t-\tau\big),$$

hence, recalling (3.2), it follows

$$0 < E_z\big(s(t)-s(\tau),t-\tau\big) \leqslant 4^{-1}\pi^{-1/2}M(t-\tau)^{-1/2}. \tag{3.9}$$

We can apply the above estimate to obtain

$$\begin{aligned}&\left|\int_0^t E_z\big(s(t)-s(\tau),t-\tau\big)\big[\varepsilon^{-1}Q\big(\hat\theta(\tau)\big)-\hat\varphi(\tau)\big]\,d\tau\right| \\ &\quad\leqslant 2^{-1}\pi^{-1/2}M^2\varepsilon^{-1}\gamma^{1/2} + 4^{-1}\pi^{1/2}M \sup_{0<\tau<\gamma}\big|\tau^{1/2}\hat\varphi(\tau)\big|.\end{aligned}$$

Hence

$$\big|(K_{\varepsilon 2}\mathbf{v})(t)\big| \leqslant \pi^{-1/2}M\varepsilon^{-1}t^{-1/2} + \pi^{-1/2}M^2\varepsilon^{-1}\gamma^{1/2} + 2^{-1}\pi^{1/2}M\|\mathbf{v}\|_\gamma. \tag{3.10}$$

Combining (3.8) and (3.10) we obtain

$$\|\mathbf{K}_\varepsilon\mathbf{v}\|_\gamma \leqslant C_0 + \pi^{1/2}\big(\varepsilon + 2^{-1}M\gamma^{1/2}\big)\|\mathbf{v}\|_\gamma,$$

where $C_0 = (1+2\pi^{-1/2}\varepsilon^{-1})M + \pi^{-1/2}(2+M\varepsilon^{-1}\gamma^{1/2})M\gamma^{1/2}$. Thus the statement of the lemma is proved with

$$\delta_1 = \frac{C_0}{1-\pi^{1/2}(\varepsilon+2^{-1}M\gamma^{1/2})} \tag{3.11}$$

provided that $\varepsilon < \pi^{-1/2}$ and $\gamma^{1/2} < 2M^{-1}(\pi^{-1/2}-\varepsilon)$. □

Lemma 3.2. *Let $\delta > 0$ and $\mathbf{v}_i \in \mathcal{B}_\gamma(\delta)$ for $i = 1,2$. Fix ε and γ as in Lemma* 3.1. *Then*

$$\|\mathbf{K}_\varepsilon\mathbf{v}_1 - \mathbf{K}_\varepsilon\mathbf{v}_2\|_\gamma \leqslant C_1\big(\pi^{1/2}\varepsilon + \gamma^{1/2} + \varepsilon^{-1}\gamma^{1/2} + \delta\gamma^{1/2}\big)\|\mathbf{v}_1 - \mathbf{v}_2\|_\gamma, \tag{3.12}$$

where the positive constant C_1 only depends upon M.

Proof. We give only a sketch. For $i = 1,2$ let $\mathbf{v}_i = (\hat\theta_i, \hat\varphi_i) \in \mathcal{B}_\gamma(\delta)$ and s_i be the corresponding functions given by formula (3.2). Consider $K_{\varepsilon 1}$: then, it is not hard to prove the estimates

$$\left|V^0\big(s_1(t),t\big)-V^0\big(s_2(t),t\big)\right|\leqslant \pi^{-1/2}M^2t^{1/2}\sup_{0<\tau<\gamma}\left|\hat\theta_1(\tau)-\hat\theta_2(\tau)\right|,$$

$$\int_0^t\left|E\big(s_1(t)-s_1(\tau),t-\tau\big)Q\big(\hat\theta_1(\tau)\big)-E\big(s_2(t)-s_2(\tau),t-\tau\big)Q\big(\hat\theta_2(\tau)\big)\right|d\tau$$
$$\leqslant\big(\pi^{-1/2}Mt^{1/2}+6^{-1}\pi^{-1/2}M^3t^{3/2}\big)\sup_{0<\tau<\gamma}\left|\hat\theta_1(\tau)-\hat\theta_2(\tau)\right|$$

and

$$\int_0^t\left|E\big(s_1(t)-s_1(\tau),t-\tau\big)\hat\varphi_1(\tau)-E\big(s_2(t)-s_2(\tau),t-\tau\big)\hat\varphi_2(\tau)\right|d\tau$$
$$\leqslant 2^{-1}\pi^{1/2}\Big(\sup_{0<\tau<\gamma}\left|\tau^{1/2}\big(\hat\varphi_1(\tau)-\hat\varphi_2(\tau)\big)\right|$$
$$+4^{-1}M^2t\sup_{0<\tau<\gamma}\left|\tau^{1/2}\hat\varphi_1(\tau)\right|\sup_{0<\tau<\gamma}\left|\hat\theta_1(\tau)-\hat\theta_2(\tau)\right|\Big).$$

Collecting now the above estimates together, we get

$$\left|(K_{\varepsilon1}\mathbf{v}_1-K_{\varepsilon1}\mathbf{v}_2)(t)\right|\leqslant C\big(\pi^{1/2}\varepsilon+\gamma^{1/2}+\gamma^{1/2}+\delta\varepsilon\gamma\big)\|\mathbf{v}_1-\mathbf{v}_2\|_\gamma,$$

where C depends upon M. Next, we consider $K_{\varepsilon2}$. We have

$$\left|V^0_y\big(s_1(t),t\big)-V^0_y\big(s_2(t),t\big)\right|\leqslant M^2\sup_{0<\tau<\gamma}\left|\hat\theta_1(\tau)-\hat\theta_2(\tau)\right|,$$

$$\int_0^t\left|E_z\big(s_1(t)-s_1(\tau),t-\tau\big)Q\big(\hat\theta_1(\tau)\big)-E_z\big(s_2(t)-s_2(\tau),t-\tau\big)Q\big(\hat\theta_2(\tau)\big)\right|d\tau$$
$$\leqslant 4^{-1}\pi^{-1/2}M\big((M+2)t^{1/2}+3^{-1}M^3t^{3/2}\big)\sup_{0<\tau<\gamma}\left|\hat\theta_1(\tau)-\hat\theta_2(\tau)\right|$$

and

$$\int_0^t\left|E_z\big(s_1(t)-s_1(\tau),t-\tau\big)\varphi_1(\tau)-E_z\big(s_2(t)-s_2(\tau),t-\tau\big)\varphi_2(\tau)\right|d\tau$$
$$\leqslant 4^{-1}\pi^{1/2}M\Big(\sup_{0<\tau<\gamma}\left|\tau^{1/2}\big(\hat\varphi_1(\tau)-\hat\varphi_2(\tau)\big)\right|$$
$$+\big(1+4M^2t\big)\sup_{0<\tau<\gamma}\left|\tau^{1/2}\hat\varphi_1(\tau)\right|\sup_{0<\tau<\gamma}\left|\hat\theta_1(\tau)-\hat\theta_2(\tau)\right|\Big).$$

We combine the above estimates to obtain

$$\left|(K_{\varepsilon2}\mathbf{v}_1-K_{\varepsilon2}\mathbf{v}_2)(t)\right|\leqslant\widetilde C\big(1+\varepsilon^{-1}+\varepsilon^{-1}\gamma^{1/2}+\varepsilon^{-1}\gamma^{3/2}+\delta+\gamma\delta\big)\|\mathbf{v}_1-\mathbf{v}_2\|_\gamma,$$

where $\widetilde C$ depends upon M. Whence the result follows since $\varepsilon<\pi^{-1/2}$ and $\gamma<\gamma_1$. □

Proposition 3.1. *Let ε be sufficiently small. Then there exist $\gamma>0$ so small and $\delta>0$ so large such that $\mathbf{K}_\varepsilon$ is a contraction mapping of the closed ball $\mathcal{B}_\gamma(\delta)$ into itself.*

Proof. Let δ_1 be given by (3.11). On taking $\gamma = O(\varepsilon^4)$ as $\varepsilon \to 0$, we get $\delta_1 = O(\varepsilon^{-1})$, hence the Lipschitz constant in (3.12) with $\delta = \delta_1$ is of order ε. Therefore, in view of the previous lemmas, there exists $\varepsilon_0 > 0$ such that, if $\varepsilon < \varepsilon_0$, $\mathbf{K}_\varepsilon$ is a contraction mapping on $\mathcal{B}_\gamma(\delta_1)$. □

We are now in a position to prove the main theorem of this section.

Proof of Theorem 3.1. Since the contraction mapping principle is applicable to Eq. (3.7), see Proposition 3.1, we can assert that the solution $(\hat{\theta}, \hat{\varphi})$ of system (3.6) of integral equations yields a solution (s, U) of the original problem $(\mathcal{P})$ in the interval $[0, \gamma]$ for γ small enough. Indeed, let $s(t)$ and $U(y,t)$ be defined by (3.2) and (3.1), respectively. Then, using the continuity of $U_{\mathrm{in}}(y)$, $\hat{\theta}(t)$ and $t^{1/2}\hat{\varphi}(t)$ and the properties of single-layer heat potential [3], we can see that s is continuously differentiable and U is continuously differentiable twice with respect to y and once with respect to t in the region Ω_γ. In addition, U is two-dimensionally continuous at the boundary of Ω_γ and has the specified datum U_{in} as its limit at $t = 0$; U_y is two-dimensionally continuous at $(s(t), t)$ for $0 < t \leqslant \gamma$ and, in particular, the flux boundary condition at $y = s(t)$ follows from (3.5) and the second of the integral equations (3.6).

In order to prove uniqueness, we recall that a solution (s, U) of the original problem has the representation (2.3) where the boundary temperature $v(t)$ must be a solution of the integral equation (2.7). We can now repeat the estimates presented in Lemmas 3.1 and 3.2 for $K_{\varepsilon 1}$ to conclude that the right-hand side of (2.7) defines a contraction map on a suitable ball of the space $\mathcal{C}^0([0,\gamma])$, endowed with the uniform norm, if $\gamma > 0$ is small enough. This easily implies that the local solution is unique. Since the argument is standard, we omit the details. □

Remark 3.1. As a matter of fact it can be shown that the solution v of the integral equation (2.7) enjoys further regularity, namely it is continuously differentiable for $t > 0$. In addition, by this property and the representation formula (2.3), it is not difficult to argue the continuity of the time derivative U_t up to the lateral boundary Γ_γ. We refer to [13, Proposition 2.1] for details concerning the proof of these facts.

4. The global solution

Under assumptions H1 and H2, Theorem 3.1 yields existence and uniqueness of a local solution of (2.1) in the time interval $[0, \gamma]$. The solution enjoys the property of Lemma 2.1, so that it can be continued to some interval $[\gamma, \gamma_1]$ with $\gamma_1 > \gamma$. Now, let (s, U) be the maximally defined solution to problem (2.1) in the interval $[0, T^*)$. We are to show that T^* is infinite, i.e. the solution of (2.1) exists in the large, if the initial thermal profile U_{in} vanishes at infinity at a rate that is exponential. The central difficulty in our proof is to argue that the same exponential decay holds for the solution *uniformly* on $[0, T^*)$. The argument requires the following lemma.

Lemma 4.1. *Let Theorem* 3.1 *hold and assume, in addition, that there exists* $\eta > 0$ *such that*

$$\sup_{y<0}\left|e^{-\eta y} U_{\mathrm{in}}(y)\right| < \infty. \tag{4.1}$$

Let (s, U) *be the maximally defined solution to problem* (2.1) *in the interval* $[0, T^*)$. *Then*

$$\sup_{y<s(t)}\left|e^{-\eta y} U_y(y,t)\right| < \infty \quad \textit{for } t \in (0, T^*).$$

Proof. We consider the representation (2.3). Differentiating with respect to y, we obtain

$$U_y(y,t)=\int_{-\infty}^{0} E_y(y-z,t)U_{\text{in}}(z)\,dz+\int_0^t E_y\big(y-s(\tau),t-\tau\big)F\big(U\big(s(\tau),\tau\big)\big)\,d\tau$$
$$+\int_0^t E_{yy}\big(y-s(\tau),t-\tau\big)U\big(s(\tau),\tau\big)\,d\tau.$$

As condition (4.1) holds and the compositions $U(s(t),t)$ and $F(U(s(t),t))$ are continuous for $t\in(0,T^*)$, we need only estimate

$$I_1=\int_{-\infty}^{0} E_y(y-z,t)e^{\eta z}\,dz,\qquad I_2=\int_0^t E_y\big(y-s(\tau),t-\tau\big)\,d\tau,$$
$$I_3=\int_0^t E_{yy}\big(y-s(\tau),t-\tau\big)\,d\tau.$$

The transformation $\lambda=\eta t^{1/2}+(y-z)(4t)^{-1/2}$ yields

$$|I_1|\leqslant(\pi t)^{-1/2}e^{\eta y+\eta^2 t}\int_{\eta t^{1/2}+y(4t)^{-1/2}}^{\infty}\big|\lambda-\eta\sqrt{t}\big|e^{-\lambda^2}\,d\lambda\leqslant C_1(t)e^{\eta y}.$$

Further, taking into account the inequality $|y-s(t)|\leqslant|y-s(\tau)|$ for all y with $y<s(t)$ and $0\leqslant\tau<t$, we have

$$|I_2|\leqslant I_{21}+I_{22},$$

where

$$I_{21}=\frac{1}{2}\int_0^t\frac{s(\tau)-s(t)}{t-\tau}E\big(y-s(t),t-\tau\big)\,d\tau,$$
$$I_{22}=\frac{1}{2}\int_0^t\frac{s(t)-y}{t-\tau}E\big(y-s(t),t-\tau\big)\,d\tau.$$

Recalling that s is Lipschitz-continuous with Lipschitz constant M, from the substitution $2\lambda=(s(t)-y)(t-\tau)^{-1/2}$ we obtain

$$I_{21}\leqslant\frac{M}{4\sqrt{\pi}}\big(s(t)-y\big)\int_{(s(t)-y)/\sqrt{4t}}^{\infty}\lambda^{-2}e^{-\lambda^2}\,d\lambda.$$

Then, using the inequality $\int_z^\infty\lambda^{-2}e^{-\lambda^2}\,d\lambda\leqslant z^{-3}e^{-z^2}/2$, $z>0$, we have for $y<2s(t)-4\eta t$

$$I_{21}\leqslant\frac{Mt^{3/2}}{(s(t)-y)^2\sqrt{\pi}}e^{-(s(t)-y)^2/4t}\leqslant\frac{Mt^{3/2}e^{-s(t)^2/4t}}{(4\eta t-s(t))^2\sqrt{\pi}}e^{\eta y}=C_{21}(t)e^{\eta y}.$$

As similar estimates hold for I_{22} and I_3, we see that there exists $C(t)$ such that, for y bounded away from $s(t)$,

$$\left|U_y(y,t)\right| \leqslant C(t)e^{\eta y}$$

and the statement follows since U_y is continuous up $y = s(t)$ for each value of $t > 0$. □

Theorem 4.1. *Let the hypotheses of Lemma* 4.1 *hold. Then the solution* (s, U) *of* (2.1) *exists in the large.*

Proof. Let (s, U) be the maximally defined solution of (2.1) in the interval $[0, T^*)$ and assume by contradiction that $T^* < \infty$. Fix $\varepsilon, \gamma \in (0, T^*)$, $\varepsilon < \gamma$. Setting $U_y = V$, then V is the solution of the initial-boundary-value problem

$$\begin{aligned}
&V_t = V_{yy}, \quad y < s(t),\ \varepsilon < t < \gamma,\\
&V(-\infty, t) = 0, \quad \varepsilon \leqslant t \leqslant \gamma,\\
&V\big(s(t), t\big) = q(t), \quad \varepsilon \leqslant t \leqslant \gamma,\\
&V(y, \varepsilon) = V_{\mathrm{in}}(y), \quad y \leqslant s(\varepsilon),
\end{aligned}$$

where $q(t) = Q(U(s(t), t))$ and $V_{\mathrm{in}}(y) = U_y(y, \varepsilon)$. We see that V has the integral representation

$$V(y,t) = \int_{-\infty}^{s(\varepsilon)} E(y-z,t)V_{\mathrm{in}}(z)\,dz + \int_{\varepsilon}^{t} E_z\big(y - s(\tau), t-\tau\big)w(\tau)\,d\tau, \tag{4.2}$$

where w is the solution of the linear integral equation

$$q(t) = \int_{-\infty}^{s(\varepsilon)} E\big(s(t)-z,t\big)V_{\mathrm{in}}(z)\,dz + \frac{1}{2}w(t) + \int_{\varepsilon}^{t} E_z\big(s(t) - s(\tau), t-\tau\big)w(\tau)\,d\tau. \tag{4.3}$$

As $|q| \leqslant M$ and $|V_{\mathrm{in}}| \leqslant \sup_{y<s(\varepsilon)} |e^{-\eta y}U_y(y,\varepsilon)| = K$ by Lemma 4.1, applying inequality (3.9) to (4.3) yields

$$\left|w(t)\right| \leqslant 2(M+K) + \frac{M}{\sqrt{\pi}}\sqrt{t-\varepsilon}\sup_{\varepsilon<\tau<t}\left|w(\tau)\right|.$$

So, if ε satisfies $\varepsilon > \max(0, T^* - 4^{-1}\pi M^{-2})$, then we get, for every $\gamma \in [\varepsilon, T^*)$,

$$\sup_{\varepsilon<\tau<\gamma}\left|w(\tau)\right| \leqslant 4(M+K).$$

This bound shows, by virtue of (4.2), that

$$\left|V(y,t)\right| \leqslant K\int_{-\infty}^{s(\varepsilon)} E(y-z,t)e^{\eta z}\,dz + 4(M+K)\int_{\varepsilon}^{t}\left|E_z\big(y-s(\tau),t-\tau\big)\right|d\tau.$$

Then, by repeating the estimates in the proof of Lemma 4.1 and letting γ go to T^*, we obtain that $e^{-\eta y}V(y,t)$ is uniformly bounded in the region $y < s(t)$, $\varepsilon < t < T^*$ for some ε, which implies the same conclusion for $U(y,t) = \int_{-\infty}^{y} V(z,t)\,dz$.

Now, choose an arbitrary initial time $t_0 \in (\varepsilon, T^*)$ and consider problem (2.1) with initial condition $U_{\mathrm{in}}(y) = U(y, t_0)$. By Theorem 3.1 we find a unique solution in some interval $[t_0, t_0 + \gamma]$; moreover, the above stated bound on U permits a uniform choice of γ with respect to t_0. This shows that the maximal solution (s, U) has an extension beyond T^*, a contradiction. Therefore $T^* = \infty$. $\square$

As a consequence of Theorem 4.1, we can assert that the global solution (s, U) satisfies

$$\sup_{y<s(t)} \left|e^{\eta(s(t)-y)} U_y(y,t)\right| < \infty \quad \text{for } t > 0, \quad \text{and} \tag{4.4}$$

$$\sup_{y<s(t)} \left|e^{\eta(s(t)-y)} U(y,t)\right| < \infty \quad \text{for } t \geqslant 0. \tag{4.5}$$

In order to study the large-time behavior of solutions, and in particular stability of the traveling wave front, the property that the left-hand sides in (4.4) and (4.5) remain bounded as $t \to \infty$ is needed [13]. We end this section by showing this result in a case of physical relevance (see Remark 4.2).

Proposition 4.1. *Let R and Q satisfy* H1*, the conditions listed in Remark* 2.1 *and*

$$Q_0 > \xi_0 R_\infty, \tag{4.6}$$

where $R_\infty = \sup_{\xi>\xi_0} R(\xi)$. Assume moreover that $U_{\mathrm{in}} \geqslant 0$ satisfies H2 *and condition* (4.1) *for $0 < \eta < R(Q_0/R_\infty)$. Then the solution (s, U) of problem* (2.1)*, given by Theorem* 4.1*, satisfies the uniform bounds*

$$\sup_{t\geqslant 0} \sup_{y<s(t)} \left|e^{\eta(s(t)-y)} U(y,t)\right| < \infty \tag{4.7}$$

and, for all $\delta > 0$,

$$\sup_{t\geqslant \delta} \sup_{y<s(t)} \left|e^{\eta(s(t)-y)} U_y(y,t)\right| < \infty. \tag{4.8}$$

Proof. Setting $x = y - s(t)$ and $\vartheta(x,t) = U(y,t)$, problem (2.1) is equivalent to (2.2). Thus we need to show that

$$\sup_{t\geqslant 0} \sup_{x<0} \left|e^{-\eta x} \vartheta(x,t)\right| < \infty \quad \text{and} \tag{4.9}$$

$$\sup_{t\geqslant \delta} \sup_{x<0} \left|e^{-\eta x} \vartheta_x(x,t)\right| < \infty. \tag{4.10}$$

For this purpose, we shall use comparison arguments. First, we note that, for $\alpha = R_\infty/2$ and $0 < \beta < Q_0$, the function

$$\underline{\vartheta}(x,t) = \frac{\beta}{2\alpha} e^{2\alpha x} + \frac{\beta}{\pi} e^{\alpha(x-\alpha t)} \int_0^\infty \frac{\alpha \sin(x\sqrt{\lambda}) - \sqrt{\lambda}\cos(x\sqrt{\lambda})}{(\lambda+\alpha^2)^2} e^{-\lambda t}\, d\lambda$$

provides us with an explicit lower bound for ϑ. Indeed, $\underline{\vartheta}$ is the solution of the linear problem

$$\underline{\vartheta}_t = \underline{\vartheta}_{xx} - 2\alpha\, \underline{\vartheta}_x, \quad x < 0,\ t > 0,$$
$$\underline{\vartheta}_x(-\infty, t) = 0, \quad t > 0,$$
$$\underline{\vartheta}_x(0, t) = \beta, \quad t > 0,$$
$$\underline{\vartheta}(x, 0) = 0, \quad x \leqslant 0,$$

and $\underline{\vartheta}_x \geqslant 0$. Hence the parabolic operator $L \equiv \partial^2/\partial x^2 - R(\vartheta(0,t))\partial/\partial x - \partial/\partial t$ satisfies $L(\vartheta - \underline{\vartheta}) = (R(\vartheta(0,t)) - 2\alpha)\,\underline{\vartheta}_x \leqslant 0$ for $x < 0$ and $t > 0$. As $\vartheta(x,0) - \underline{\vartheta}(x,0) = U_{\rm in}(x) \geqslant 0$, $\vartheta(-\infty,t) - \underline{\vartheta}(-\infty,t) = 0$ and $\vartheta_x(0,t) - \underline{\vartheta}_x(0,t) = Q(\vartheta(0,t)) - \beta \geqslant Q_0 - \beta > 0$, the minimum principle yields $\vartheta \geqslant \underline{\vartheta}$ through the region $x \leqslant 0$, $t \geqslant 0$.

Next we compare ϑ_x with the functions $\pm Ce^{\eta x}$, where $C > 0$ will be chosen later and $0 < \eta < R(\beta/R_\infty)$. Since $\underline{\vartheta}(0,t)$ is increasing and tends to $\beta/2\alpha$ as $t \to \infty$, this means that there exists a $\bar{t} > 0$ such that $\underline{\vartheta}(0,\bar{t}) = R^{-1}(\eta)$. Therefore, for all $t \geqslant \bar{t}$, $R(\vartheta(0,t)) \geqslant R(\underline{\vartheta}(0,t)) \geqslant R(\underline{\vartheta}(0,\bar{t})) = \eta$ whence it follows, for $x < 0$ and $t > \bar{t}$, $L(v_-) = C\eta(R(\vartheta(0,t)) - \eta)e^{\eta x} \geqslant 0$, where $v_- = \vartheta_x - Ce^{\eta x}$. We further know that there exists $\overline{C} > 0$ such that $|\vartheta_x(x,\bar{t})| \leqslant \overline{C}e^{\eta x}$ for $x \leqslant 0$. Hence, taking $C = \max(\overline{C}, \sup_{\xi > 0} Q(\xi))$, at the boundary of the region $x < 0$, $t > \bar{t}$ we have $v_-(x,\bar{t}) = \vartheta_x(x,\bar{t}) - Ce^{\eta x} \leqslant 0$, $v_-(0,t) = Q(\vartheta(0,t)) - C \leqslant 0$ and $v_-(-\infty,t) = 0$. Consequently, we see that $v_- \leqslant 0$ for $x \leqslant 0$, $t \geqslant \bar{t}$. In a similar manner we get $v_+ = \vartheta_x + Ce^{\eta x} \geqslant 0$ and the bound (4.10) follows. The remaining estimate (4.9) is obtained by integration. □

Remark 4.1. More generally, if we assume $U_{\rm in} \geqslant -m$ for some $m \geqslant 0$, the above uniform bounds hold true provided condition (4.6) is replaced by $Q_0 > (m + \xi_0)R_\infty$ and η satisfies $0 < \eta < R(Q_0/R_\infty - m)$.

Remark 4.2. Coming back to physical units, the key inequality (4.6) becomes

$$I > cr_\infty(T_0 - T_1),$$

where $r_\infty = \sup_{T > T_0} r(T)$. Thus in practical applications condition (4.6) is satisfied by selecting suitable values for the system operational parameters (ambient temperature T_1 and radiant flux intensity I): this is the case, for example, when the burning surface is irradiated by an external source whose power intensity is large enough.

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